THE TEACHER'S OUTFIT IN MATHEMATICS.

An important element in a teacher's outfit is a keen appreciation of the educational value of his subject. Such an appreciation inspires him to set a high ideal for himself and his pupil, and compels the constant broadening of his knowledge; this alone can beget within him a just estimate of the dignity and importance of his calling, a consciousness of his own grave responsibility. This thought should be especially considered by those who are preparing to teach mathematics, for there is no subject the first feeling toward which depends so largely on the enthusiasm and personal charm of the teacher, the mastery and enjoyment of which are so completely a question of the teacher's power to interpret the subject consistently as a science, and yet to adapt it to the growing powers of the pupil.

So commonplace and simple are the facts of arithmetic and algebra, so tempting is it to rely on memory rather than on logical processes, to make the development of the subject a mere mechanical work of imitation and acceptance of authority rather than an exercise in independent thinking, that the teacher is led to feel that he has before him no difficult task of presentation, that there is no need of loyalty to the nature of the subject or of close definition of the mind and heart qualities that are the result of successful work. Most of all, in primary and secondary instruction, a teacher should realize that he is determining, perhaps permanently, the pupil's attitude toward a whole range of intellectual activity, that upon him depends whether the subject shall arouse active interest or the greatest aversion, that it is his to decide whether the pupil shall assimilate knowledge, develop power, and gain the elements of scientific culture, or have his youthful spirit continually baffled and broken by defeat.

The study of mathematics has an educational value as contributing knowledge, discipline, and culture. Since mathematical

science is the instrument of indirect measurement, or measurement by computation, it has a knowledge value which is illustrated in all vocations of life. By it the architect builds our homes and cities; by it the engineer constructs our railways, bridges and tunnels; the surveyor maps a farm or a continent; the astronomer measures the earth, sun, moon and planets, determines their paths, marks their seasons, and foretells their positions; by its aid the mariner courses the ocean and brings all nations into friendly and helpful intercourse. Without it civilization would be an impossibility, and commerce a dream.

To the disciplinary value of mathematical studies, Plato bore witness in the inscription over his porch: "Let no one who is unacquainted with geometry enter here." The concepts of number and space which are the simplest of all abstract notions are combined by the simplest logical processes; hence, mathematical studies afford the easiest and most natural introduction to the severer abstract studies. A clear conception of data, a rigorous exactness in reasoning, the detection and correction of every error help to form the habit of insisting upon adequate mastery, whatever may be the subject.

The Greeks recognized the culture value of mathematical study, though they did not fully appreciate its knowledge value. Rigorously exact in reasoning, they felt the beauty of a demonstration. The curves which they invented to solve two of the three most celebrated problems of antiquity bear the poetic names, cissoid and chonchoid. Their geometry was embodied only in the perfection of their architecture; for the inventors of the beautiful science would keep the artistic product of their imagination and reason from too sordid uses. To study geometry, according to Plato, was to think the Deity's thoughts after him, for "He geometrizes continually." By its magic touch, mathematics has opened wide the door of the heavens and transformed the fiction of astrology into the grandest of sciences. More than any other science, has the study of mathematical truth embodied in the universe enlarged and liberalized the mind and character of man. Hostile to narrowness and conceit,

astronomy cultivates humility, candor, and the childlike spirit. "This science," says Dr. Burr, "is worth more than all the fictions and poems of the world as a judicious cultivator of the imagination and corrector of insipidity and barrenness of character. It is a poem as well as a science, the best example we have of polished completeness in a science, and the noblest specimen we have of an epic poem."

For convenience we may divide secondary mathematical studies into two groups, the first group including arithmetic and concrete geometry, the second, algebra and demonstrative geometry. In the first group, the practical and knowledge value of the studies should be emphasized; in the second, their power for discipline should be the controlling consideration. In both groups the method of instruction, the atmosphere that is thrown around the subject, and above all the mental quality and personal character of the teacher should make for that patience in truth-seeking, that eagerness to attain demonstrated knowledge, that passion for perfection which are fundamental in the culture of a man of science.

The student of arithmetic and concrete geometry has a conscious desire to know rather than to prove. Hence instruction should be objective and practical, but should lead to education in its true sense. By questions the teacher should draw out the pupil's ideas and powers. "The teacher must question the subject into his pupils and then he must question it out of them." Under careful direction, by trial, measurement, construction and superposition, the pupil should discover the simpler truths of arithmetic and geometry in logical order and be required to apply them to many of their manifold practical uses. The teacher's skill in this leadership will depend upon his tact, his complete mastery of the subject, and his sympathy with the growing mind. Such an empirical knowledge of the subject and of its useful and varied applications awakens an interest and a questioning attitude in the pupil. He glories in his power to do work at a long range, to measure the height of a tree without climbing it, the breadth of a stream without

crossing it, or the volume of a solid without applying a unit of volume.

Accordingly to attain highest success in arithmetic and concrete geometry, the teacher must not only be conversant with algebra, geometry and trigonometry as sciences, but he must also have a living interest in them, as instruments of indirect measurement. Again he should know and be able to apply the psychological principles, obedience to which is the secret of success. His plan for presenting each subject should be well matured, systematic and logical, yet artfully adapted to the changing conditions of the child mind ; objects in and about the school room should suggest the first illustrations, applications, and problems of concrete geometry. Such a course makes the pupil familiar with the individual notions of form and relations before he comes to deal with the general ideal notions of demonstrative geometry.

The teacher's function lies in aiding the pupil "to get clear and definite individual notions; and to pass from them to equally clear and definite general notions ; to do this by the exercise of his own powers so that knowledge may be assimilated and power developed." Thus, in arithmetic we first get clear ideas of individual numbers which we denote by figures. Later, however, in arithmetical studies we should gain general notions of arithmetical numbers which should be denoted by letters. In conformity with this principle also, the beginner in algebra should first gain clear and definite notions of individual real algebraic numbers, which have quality as well as an arithmetical value and which require for expression, figures to denote arithmetical value, and the sign + or — to indicate quality. Not until the laws and methods of performing the fundamental operations on individual algebraic numbers have become familiar should the general notion of algebraic number and its literal symbol be introduced.

Though arithmetic and concrete geometry are of the greatest practical value as knowledge, it must be remembered that they are not merely ends in themselves, but are introductions to

algebra and demonstrative geometry, that even in these preliminary studies every possible encouragement should be given to the fostering of the true scientific spirit. In both arithmetic and concrete geometry, the learner should employ the literal notation and linear equations as aids in solving problems, also give the logical proofs of the simpler and more common principles. An admiration for the wonder-working powers of the equation, a taste for the fascinating problems of geometry makes easy the transition to the second group of studies.

Algebra and geometry should be demonstrative. The prime object of these studies is to develop and discipline the reasoning powers. If they fail to accomplish this purpose, they fail utterly. Their knowledge value to the majority of students would not at all justify the usual expenditure of time and energy upon them. Euclid's early reduction of geometry to a scientific form has made it the study, which, above all others, teaches rigorous demonstration and logical order. The general aims and methods of a course in geometry are admirably set forth in the report of the Committee of Ten, as also in the February number of the SCHOOL REVIEW.

It would add greatly to the interests of sound mathematical learning, if algebra were pursued as a strictly demonstrative science. For purposes of advanced study, algebra is more important than geometry; in fact, the higher geometric methods are based entirely upon it. No mathematical instrument is so generally employed as the equation, and none is more carelessly and illogically treated in the class room. This is the more to be regretted since the interest of the pupil in algebra as well as his development of reasoning power would be greatly increased by a close and scientific presentation of the subject.

To promote the teaching of algebra strictly as a science, we offer the following suggestions:

1. A teacher of algebra should have a well defined idea of the essential nature of his subject. Auxiliaries and incidentals should not be mistaken for the central thought. The literal notation and the system of signs and symbols employed in

algebra do not constitute its distinguishing characteristic. In arithmetic we have much the same system of signs, and arithmetical studies should render the student familiar with the literal notation. Moreover, algebra was cultivated long before the notation of letters and signs was invented. The Ahmes Papyrus, written earlier than 1700 B. C., and founded on an older work, believed by Birch to date back as far as 3400 B. C., contains the beginnings of algebra in the solution of equations of one unknown. In this work the unknown is called "hau" or heap, and the simple equation $\frac{1}{7}x + x = 19$ is awkwardly expressed thus: "heap, its $\frac{1}{7}$, its whole, it makes 19." Beginning at this early date the history of algebra is the history of the equation. The notation of algebra, including symbols of operation, relation, abbreviation, and quantity was invented to secure clearness and facility in the statement, transformation and solution of equations. The number of algebra was conceived in the effort to interpret such results as -4, $-\sqrt{-3}$, which were obtained as roots of equations. Hence, the essential thing in the study of algebra is the equation, its nature and properties, the methods of solving it, and its use as a mathematical instrument. Algebraic number, its notation, the laws of combination, the fundamental operations, factoring, etc., should each and all be viewed simply as aids to the study of the equation.

Some knowledge of the nature and usefulness of the equation cannot fail to arouse the interest of the pupil in the science of algebra. Hence, if this work has not already been done in arithmetic, an introduction to algebra should explain the literal notation as applied to arithmetical number, give a clear idea of the nature of linear equations with arithmetical roots, and fully illustrate the beauty and power of this mathematical instrument by the solution of numerous arithmetical problems. This will prepare the student for the scientific study of the equation and the enlarged concept of number to which that study leads.

2. The concept of algebraic number and its notation should be clear and definite in the teacher's mind. An arithmetical number answers the one question "How many?" an algebraic

number answers the two questions "How many?" and "Of what quality?" An arithmetical number has no quality; regarding such number as positive leads to confusion of ideas. Again, the quality of an algebraic number should not be called its sign; for thus a concept is confounded with its symbol, and the learner is led to think that the quality of a number is always denoted by a sign. It cannot be too strongly emphasized that a letter denotes both the quality and the arithmetical value of an algebraic number. With such an interpretation, the learner easily comprehends the meaning of general formulas and appreciates general discussions; as a student of analytic geometry, he will not limit the point (x,y) to the first quadrant nor the point $(-x,-y)$ to the third; nor will he regard the line $y=mx+b$ as including only those having a positive slope and intercept. Calling a term positive or negative according to the quality of its numerical coefficient inevitably misleads the student; unconsciously he comes to regard such a term as $+\ 2x$, as denoting a positive number, and $-\ 3y$ a negative number.

In arithmetic the signs $+$ and $-$ denote operations, but in algebra they more generally denote the quality of numbers. Thus, in a polynomial which is not a general formula, the sign before each term is best regarded as the sign of the quality of its numerical coefficient, the $+$ sign of operation being understood between each two terms. Every polynomial is thus regarded as a sum, and theorems proved for sums are general for all polynomials. Hence, the law of signs in arithmetic is the law of signs of operation, while the law of signs in algebra is the law of signs of quality. The proof of the arithmetical law should not be mistaken or misused for that of the algebraic law. Thus, if 5, 4 and 3 denote arithmetical numbers, we can easily prove that $(5-4)\times 3\equiv 15-12$, but this does not prove that $(-4)(+3)\equiv -12$. To prove the latter identity, we need to give the meaning of quality to the signs $+$ and $-$, and a more general signification to the operation of multiplication.

3. The successive steps in the evolution of the general concepts of number, operations, etc., should be recognized and care-

fully considered. Many concepts in mathematics spring from a seed, as it were, and have a natural growth. This progressive development of the subject should be emphasized; for it is essential to clear thinking, promotes interest, and prepares for scientific, untrammelled investigation in any sphere. To illustrate, let us consider the evolution of the idea of number. First, by counting we gain the idea of an arithmetical whole number; next, by dividing a unit into equal parts we reach the notion of a fraction; and in computing the diagonal of a square, we form the concept of an incommensurable number as the limit of a varying commensurable number. This in arithmetic. Later by study of linear equations in algebra, we gain the concept of real algebraic number, or the simplest number having the element of quality. Then, farther on in the study of quadratic and higher equations, or in the attempt to express by numbers both the magnitude and direction of forces, we gain the general concept of algebraic number, which is represented graphically by a vector, or directed line.

This enlarging concept of number naturally leads to a broader view of operations. Thus, the operation of multiplying by an arithmetical whole number is very simple and intimately connected with addition. But when the multiplier is a fraction or an algebraic number, our first idea of multiplication is inadequate, and multiplication must be defined somewhat as follows: To multiply one number by another is to treat the first in the same way that we would treat unity, or $+1$, to obtain the second. This definition removes all mystery from the idea of multiplying by an algebraic number, whether it is real, imaginary or complex. It also yields a very simple proof of the law of quality.

Again, when a and b denote arithmetical whole numbers, the fraction $\frac{a}{b}$ may be defined as denoting a of the b equal parts of unity. But when a and b denote fractional or algebraic numbers, this definition fails and must give place to the broader view of a fraction as simply an indicated quotient. Hence, in algebra any indicated quotient should be regarded as a fraction, and

the properties of fractions established in division. This simplifies the proofs and methods in division, and secures conciseness and clearness in the subjects of fractions and ratios.

4. The order of topics in algebra should be determined mainly by two considerations:

First, to present the subject as a whole in a scientific and attractive form.

Second, to introduce as early as possible the equation, and to show how each topic is auxiliary to the study of the equation. For example, a logical order requires that the distributive law of multiplication be proved before it is used in the addition of products, and that the law of quality be proved before it is used in the subtraction of products. Involution and the binomial theorem for positive integral exponents should come early in the course on the ground that a power is the simplest product, that it is governed by the simplest laws, and that its study is the best introduction to the work of discovering and applying the laws which govern any expression. Evolution naturally follows involution as its inverse operation, and precedes the subject of factoring as one of its simplest cases.

Since factoring is fundamental in the study and solution of equations, the subject of quadratic and fractional equations should follow immediately the application of factoring to fractions. At the outset the problem of factoring the expression $f(x)$, should be identified with that of solving the equation $f(x)=0$. This emphasizes the use of factoring and leads to the solution of many quadratic and higher equations by inspection. Since the theory of the equivalency of equations of one unknown is simpler than that of systems of equations, the subject of quadratic equations with one unknown should precede the study of systems of linear equations. This order allows the principles of the equivalency of equations of one unknown to be made familiar in the study of quadratic and the simpler forms of higher equations, before the subject of equivalent systems is considered.

5. The teacher's outfit must include a thorough knowledge

of the theory of the equivalency of equations. An identity is an equality whose members are identical expressions. An equation is an equality whose members are non-identical expressions, one or both of which contains a letter or letters. Two equations are said to be *equivalent* when the roots of each include all the roots of the other. Hence, in solving an equation, we need to know what operations upon it will lead to an equivalent equation and what operations will not; that is, we need to understand the theory of the equivalency of equations. To illustrate the importance of this theory, let us consider the equation,

$$\frac{2}{x^2 - 1} - 1 = \frac{1}{x(x-1)}. \qquad (1)$$

By the theory of equivalency of equations we know that to obtain an integral equation equivalent to (1), we must multiply (1) by the simplest unknown factor that will clear it of fractions. Hence, transposing and adding the fractions in (1), we obtain, after simplification the equation

$$\frac{1}{(x+1)x} = 1, \qquad (2)$$

which we know to be equivalent to (1).

Multiplying (2) by $(x+1)x$ we obtain

$$x^2 + x = 1, \text{ or } x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}. \qquad (3)$$

Equations (2) and (3) are equivalent; for 0 and -1 are the only roots that could be introduced by multiplying (2) by $(x+1)x$, and (3) has neither of these numbers as a root.

If to clear equation (1) of fractions we should multiply its members by $x(x+1)(x-1)$ we would obtain the equation

$$x^3 - 2x + 1 = 0, \qquad (4)$$

of which the roots are 1 and $-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}$. Here the root 1 was introduced by multiplying equation (1) by the factor $x - 1$, which, as was seen above, is a factor unnecessary to clear (1) of fractions.

Again, take the equation

$$\sqrt{2x+8} + 2\sqrt{x+5} = 2, \qquad (1)$$

in which, for the sake of definiteness, we regard the radical sign

as denoting the positive square root. Transposing and squaring we obtain

$$x+8=4\sqrt{x+5}. \tag{2}$$

Squaring (2) and simplifying, we obtain

$$x^2-16=0, \text{ or } x=\pm 4. \tag{3}$$

By a principle of the equivalency of equations, we know that by squaring the two members of an equation, new roots are in general introduced, but no root will be lost. Hence, if (1) has any root, it must be $+4$ or -4, that is, one of the roots of equation (3). By trial we find that $x=+4$ does not satisfy equation (1), while $x=-4$ does. Hence -4 is a root of (1) while $+4$ is not; the root $+4$ was introduced by squaring (1) or (2). Since $x=+4$ satisfies (2), the root $+4$ was introduced by squaring (1).

It should be noted that if we use both the positive and the negative values of $\sqrt{2x+8}$ and $\sqrt{x+5}$ in all their combinations, we obtain in addition to (1) the three equations:

$$-\sqrt{2x+8}+2\sqrt{x+5}=2, \tag{4}$$

$$\sqrt{2x+8}-2\sqrt{x+5}=2, \tag{5}$$

$$-\sqrt{2x+8}-2\sqrt{x+5}=2. \tag{6}$$

Equation (3) could be obtained from (4), (5) or (6), in the same way it was from (1). Hence, the roots of (3) include the roots of (4), (5), or (6). By trial we find that both $+4$ and -4 are roots of (4), but that neither is a root of (5) or (6). Hence, no root would be introduced by squaring (4) to free it of radicals. Equations (5) and (6) are impossible; that is, no value of x real, imaginary, or complex will render either an identity. Two roots will be introduced by freeing either (5) or (6) of radicals.

The theory of the equivalency of systems of equations is no less important than that of equations. The loci of equations are great aids to the study of equations and admirably illustrate the idea of the equivalency of systems. Hence, the progressive teacher should be familiar with graphic algebra and the elements of analytic geometry.

Until the theory of the equivalency of equations and systems of equations is applied in their study and solution, algebra will be taught and studied more as an art than as a science; and both teacher and pupil will fail to gain that genuine interest and that strong intellectual incentive which spring from clear thinking and the sense of intellectual mastery.

6. Again, the well-equipped teacher should have a thorough knowledge of the theory of limits and of its numerous applications to geometry and analysis. Demonstrations which involve limits should be rigorous. The magic power of this theory as seen in the solution of the abstruse problems of the higher mathematics cannot fail to excite wonder and interest.

For the sake of his own culture, and that of his pupils, a teacher's outfit should include a familiarity with the history of mathematics. This history not only reveals our great inheritance, but also suggests how we may increase our store. It shows how the grandest of sciences has been gradually evolved from the simplest beginnings; and teaches the lesson of openness and candor of mind to the seeker for truth. It warns against too hasty generalization, by showing how a whole range of thought may be obscured by overlooking or ignoring an exception to a general principle. It teaches how a trifling incident or even a gross error may finally lead to the enlargement and enrichment of the realm of truth; and shows how the profoundest knowledge comes by inspiration to the diligent and open-souled.

By historical remarks and anecdotes, the teacher can often greatly increase the interest of his pupils, quicken their appreciation of the wonderful inventions and discoveries of the past, and arouse new desires and scholarly ambitions. The pupil should realize that the simple and perfect "Arabic notation" which he has learned in a month, represents the combined efforts of centuries; that the perfection of this system waited long for some one to invent that Columbus egg, the zero. After a pupil has bisected an angle, he will be surprised to hear of the futile efforts to solve, with the rule and compass, the apparently simple

problem of trisecting an angle. When he has doubled a square, he will be interested in the traditions concerning the celebrated problem of the duplication of the cube, and in the many fruitless attempts to solve it by elementary geometry. Thus he will learn how a science is built up by the contributions of thinker after thinker, and will gain his first suggestions of scientific progress and its cost in human effort and sacrifice.

Where the formal and abstract are so prominent, the human and personal in a study must center about the teacher, and he must prepare to meet such demands upon him. The charm and force of his personality should be rich resource against the temptation to call the subject mechanical and uninteresting. The human qualities of mind and heart that make science possible should be the real attainments striven for by teacher and pupil alike. More than any other teacher, the instructor in mathematics can impress upon his pupils and develop in them the spirit and mental habits of the man of science. Hence, teacher and pupil must unite zealously to seek the true thing for the sake of its truthfulness; they must resolve together to accept nothing without proof; they must strive for artistic perfection of form and for an appreciation of the beauty of demonstration. And the teacher's hope in it all is that he may increase the intellectual honesty, critical power, and taste of the future citizen, and perhaps give early impulse and inspiration to the future man of science.

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